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by

M. S. Steinberg

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Earl E. Hays, Department Chairman
Department of Geology & Geophysics

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Resonance Scattering of Sound by a Small Gaseous Object of Arbitrary Form*

M. S. STEINBERG

Smith College, Northampton, Massachusetts 01060

and

Woods Hole Oceanographic Institution, Woods Hole, Massachusetts 02543

The acoustic scattering amplitude for a dilute gaseous object of arbitrary form in a liquid medium is determined self-consistently for the case of linear dimension small as compared to the wavelength. A scatterer of bulk modulus β_o and density ρ_o in a medium of bulk modulus β and density ρ may exhibit monopole resonance scattering at a frequency $\omega_r=(4\pi\beta_o C/\rho V)^{\frac{1}{2}}$, in which V is the volume and C is the capacitance in electrostatic units of a conducting replica of the scatterer. The criterion for occurrence of the resonance phenomenon is $3F\beta_o/\beta\ll 1$, in which the shape factor $F=4\pi C^3/3V\geqslant 1$ is minimum for a sphere. Dipole scattering is given in terms of the polarizability dyadic of a nonconducting replica of dielectric constant ρ/ρ_o, and is negligibly small in a neighborhood of the resonance frequency.

INTRODUCTION

THE acoustic scattering cross section for a spherical gas bubble in a liquid medium at sufficiently low ambient pressure exhibits a prominent maximum at a frequency for which the radius of the bubble is much smaller than the wavelength of the exciting field. With β denoting the bulk modulus and ρ the density of the medium, and with β_0 and ρ_0 denoting these properties for a scatterer of radius a, the peak frequency is[1]

$$\omega_r=(3\beta_0/\rho a^2)^{\frac{1}{2}}. \tag{1}$$

Such a response to very long wavelength excitation resembles that of a relatively massless spring working against a relatively incompressible load. It is usually called resonance scattering, and ω_r the resonance frequency. This paper extends the theory of resonance acoustic scattering to encompass nonspherical scatterers. Dissipation is neglected.

Rayleigh has considered scattering by a small fluid object of arbitrary form which offers only weak acoustic contrast to the surrounding medium.[2] For a plane wave exciting pressure of unit amplitude, the scattering amplitude due to an obstacle of volume V was found to be

$$f=[(\beta/\beta_0-1)k^2-(\rho/\rho_0-1)\mathbf{k}_s\cdot\mathbf{k}_e]V/4\pi, \tag{2}$$

in which $\mathbf{k}_s$ and $\mathbf{k}_e$ are the scattered and exciting wave vectors, respectively. The derivation assumes $|\beta/\beta_0-1|\ll 1$, $|\rho/\rho_0-1|\ll 1$ and $k\rightarrow 0$, which conditions assure weak scattering and permit approximating the pressure in the interior of the scatterer by the exciting pressure. Our purpose is to derive an analogous result for the case of a dilute gaseous scatterer in a liquid medium.

In Sec. I, the forms of the scattering amplitude and of a pair of integral equations that determine the acoustic field at the scatterer are approximated to lowest consistent order in the ratio of linear dimension to wavelength. In Sec. II, the latter equations are separated into two simpler pairs which independently determine the monopole and dipole coefficients of the scattering amplitude, and the coefficients are evaluated by electrostatic analog arguments that are self-consistent for an infinitely dilute scatterer ($\rho_0/\rho\rightarrow 0$). The solution is assumed valid for a scatterer of finite but sufficiently small density. It is found that resonance scattering may occur at the frequency of unstimulated monopole

* Contribution No. 1388 from the Woods Hole Oceanographic Institution.

[1] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Co., New York, 1953), Vol. 2, p. 1498.

[2] J. W. Strutt Lord Rayleigh, *The Theory of Sound* (Dover Publications, Inc., New York, 1945), Vol. 2, p. 149.

radiation computed by Strasberg,[3] and that the axis of dipole scattering need not be parallel to the exciting wave vector. The conditions under which resonance scattering actually occurs are discussed in Sec. III. It is concluded that the ambient pressure must be sufficiently low and that the shape and possibly the material of the scatterer must conform to certain restrictions.

I. SMALL SCATTERER APPROXIMATION

For a steady state in which the acoustic field varies as $\exp(-i\omega t)$, the excess pressure p_0 in the interior of the scatterer is governed by

$$p_0(\mathbf{r})=\oint\frac{e^{ik_0R}}{4\pi R}\nabla' p_0(\mathbf{r}')\cdot d\mathbf{S}'-\oint p_0(\mathbf{r}')\nabla'\frac{e^{ik_0R}}{4\pi R}\cdot d\mathbf{S}',\quad (3)$$

in which $k_0=\omega(\rho_0/\beta_0)^{\frac{1}{2}}$ and $R=|\mathbf{r}-\mathbf{r}'|$, the integration extending over the surface σ of the scatterer.[4] The net exterior pressure p due to exciting pressure p_e is determined by

$$p(\mathbf{r})=p_e(\mathbf{r})-\oint\frac{e^{ikR}}{4\pi R}\nabla' p(\mathbf{r}')\cdot d\mathbf{S}'+\oint p(\mathbf{r}')\nabla'\frac{e^{ikR}}{4\pi R}\cdot d\mathbf{S}',\quad (4)$$

with $k=\omega(\rho/\beta)^{\frac{1}{2}}$. Solutions of Eqs. 3 and 4 also satisfy

$$\nabla^2 p_0+k_0^2 p_0=0 \quad (5)$$

and

$$\nabla^2 p+k^2 p=0. \quad (6)$$

The interior and exterior acoustic fields are joined by the boundary conditions

$$p_0=p \quad \text{on} \quad \sigma \quad (7)$$

for the pressure and

$$\rho\partial p_0/\partial n=\rho_0\partial p/\partial n \quad \text{on} \quad \sigma \quad (8)$$

for the normal component of the pressure gradient. The internal and external wave numbers are related by $k_0=mk$, in which $m=(\rho_0\beta/\rho\beta_0)^{\frac{1}{2}}$ is the index of refraction.

Equation 4 incorporates the condition that the scattered wave be outgoing in the wave zone. The distance R from a point on σ to a field point has the asymptotic value $R\sim r-\mathbf{r}'\cdot\mathbf{r}/r$ at a great distance from an origin situated near σ. Thus we have $kR\sim kr-\mathbf{k}_s\cdot\mathbf{r}'$, in which $\mathbf{k}_s=k\mathbf{r}/r$ is the scattered wave vector. The scattered pressure, $p_s=p-p_e$, has the asymptotic form

$$p_s\sim fe^{ikr}/r, \quad (9)$$

in which the scattering amplitude f is given by

$$f=-\oint e^{-i\mathbf{k}_s\cdot\mathbf{r}'}[\nabla' p(\mathbf{r}')+ip(\mathbf{r}')\mathbf{k}_s]\cdot\frac{d\mathbf{S}'}{4\pi}. \quad (10)$$

It will be convenient to transform this surface integral into an integral over the volume V of the scatterer. Employing Eqs. 5–8, we obtain

$$f=\int e^{-i\mathbf{k}_s\cdot\mathbf{r}'}\left[\left(\frac{\beta}{\beta_0}-1\right)k^2 p_0(\mathbf{r}')+i\left(\frac{\rho}{\rho_0}-1\right)\mathbf{k}_s\cdot\nabla' p_0(\mathbf{r}')\right]\frac{dV'}{4\pi}. \quad (11)$$

To evaluate f, it is necessary to solve Eqs. 3 and 4 for the acoustic field at the scatterer and substitute into Eq. 11. To obtain a general result, it is sufficient to do this for a plane wave exciting pressure of unit amplitude,

$$p_e=e^{i\mathbf{k}_e\cdot\mathbf{r}}, \quad (12)$$

in which $\mathbf{k}_e$ is a vector of magnitude k. We attempt a solution only for cases in which the linear extension of the scatterer is small as compared to the wavelength $2\pi/k$ of the exciting field. Let L denote the maximum linear dimension of the scatterer, measured from an origin of coordinates which remains to be chosen. Since $|\mathbf{k}_s\cdot\mathbf{r}'|\leqslant kL$, we assume that the scattering amplitude for sufficiently small kL is adequately approximated by

$$f=\left(\frac{\beta}{\beta_0}-1\right)k^2\int\frac{p_0 dV}{4\pi}+i\left(\frac{\rho}{\rho_0}-1\right)\mathbf{k}_s\cdot\int\frac{\nabla p_0 dV}{4\pi}. \quad (13)$$

Rayleigh's result, Eq. 2, is obtained if p_0 and ∇p_0 are assigned the limiting values of p_e and ∇p_e for $k\rightarrow 0$—that is, if we put $p_0=1$ and $\nabla p_0=i\mathbf{k}_e$ in Eq. 13. To obtain a self-consistent approximation, we first expand the exponential functions of Eqs. 3, 4, and 12 in powers of k_0R, kR and $\mathbf{k}_e\cdot\mathbf{r}$. Since $k_0R\leqslant 2mkL$, $kR\leqslant 2kL$ and $|\mathbf{k}_e\cdot\mathbf{r}|\leqslant kL$ for field points within and just outside σ, it is expected that these series converge rapidly for such points when kL is sufficiently small. Thus we assume that truncating the series yields an adequate approximate description of the acoustic field at a small scatterer. Retaining terms through first order in the small quantity kL provides the lowest-order approximation that guarantees that the dipole term of Eq. 13 shall be nonvanishing. Thus the acoustic pressure at the scat-

[3] M. Strasberg, J. Acoust. Soc. Am. 25, 536–537 (1953).
[4] Ref. 1, p. 1068ff.

terer in the approximation of Eq. 13 is governed by

$$p_e = 1 + i\mathbf{k}_e\cdot\mathbf{r}, \tag{14}$$

$$p_0(\mathbf{r}) = \oint \frac{1}{4\pi R}\nabla' p_0(\mathbf{r}')\cdot d\mathbf{S}' - \oint p_0(\mathbf{r}')\nabla'\frac{1}{4\pi R}\cdot d\mathbf{S}' - ik_0 q_0 \tag{15}$$

and

$$p(\mathbf{r}) = p_e(\mathbf{r}) - \oint \frac{1}{4\pi R}\nabla' p(\mathbf{r}')\cdot d\mathbf{S}' + \oint p(\mathbf{r}')\nabla'\frac{1}{4\pi R}\cdot d\mathbf{S}' + ikq, \tag{16}$$

in which we have defined

$$q_0 = -\oint \nabla p_0\cdot d\mathbf{S}/4\pi \tag{17}$$

and

$$q = -\oint \nabla p\cdot d\mathbf{S}/4\pi. \tag{18}$$

II. ELECTROSTATIC ANALOG SOLUTION

Solutions of Eqs. 15 and 16 also satisfy Laplace's equation in the interior and exterior domains. This suggests a method of analysis in which the acoustic pressure is regarded formally as the potential in an electrostatic analog problem, restricted by the condition that electric charge may reside only on the surface σ. The quantity q is the net charge in electrostatic units enclosed by a surface situated just outside σ. It is expressible, upon employing Eqs. 8, 5, and 7, to transform the surface integral into an integral over the volume of the scatterer, as

$$q = \frac{\beta}{\beta_0}k^2\int\frac{p_0 dV}{4\pi}. \tag{19}$$

But the quantity q_0, which may be transformed into

$$q_0 = m^2k^2\int\frac{p_0 dV}{4\pi}, \tag{20}$$

is the net charge in the interior of the analog scatterer. An electrostatic analog problem cannot be specified for nonvanishing q_0. Note, however, that $q_0/q = m^2\beta_0/\beta = \rho_0/\rho$ and therefore that $q_0/q \to 0$ for an infinitely dilute gaseous scatterer in a liquid medium ($\rho_0/\rho \to 0$). Thus we may continue to seek an electrostatic analog solution for this limiting case. We assume that such a solution will be a valid approximation for a scatterer of finite but sufficiently small density. The electrostatic analog of the scatterer, according to boundary condition (Eq. 8), is a nonconducting replica of the actual scatterer having dielectric constant ρ/ρ_0 relative to the external medium. The polarization of such a material is $-(\rho/\rho_0 - 1)\nabla p_0/4\pi$ in electrostatic units, and the quantity

$$\boldsymbol{\mu} = -\left(\frac{\rho}{\rho_0} - 1\right)\int\frac{\nabla p_0 dV}{4\pi} \tag{21}$$

is the net electric dipole moment of the replica. Eqs. 19 and 21 may be employed to write Eq. 13 in the simple form

$$f = (1 - \beta_0/\beta)q - i\mathbf{k}_s\cdot\boldsymbol{\mu}, \tag{22}$$

in which q and $\boldsymbol{\mu}$ are monopole and dipole scattering coefficients, respectively.

We must solve Eqs. 15 and 16 for q and $\boldsymbol{\mu}$ in the limit $\rho_0/\rho \to 0$, which is also the limit $\beta_0/\beta \to 0$. For this purpose, it is convenient to separate the acoustic field at the scatterer into two partial fields, driven independently by the monopole and dipole parts of the exciting pressure. Thus let the partial interior pressure ψ_0 and exterior pressure ψ be determined by

$$\psi_0(\mathbf{r}) = -ik_0 q_0 + \oint \frac{1}{4\pi R}\nabla'\psi_0(r')\cdot d\mathbf{S}' - \oint \psi_0(\mathbf{r}')\nabla'\frac{1}{4\pi R}\cdot d\mathbf{S}' \tag{23}$$

and

$$\psi(\mathbf{r}) = 1 + ikq - \oint \frac{1}{4\pi R}\nabla'\psi(r')\cdot d\mathbf{S}' + \oint \psi(\mathbf{r}')\nabla'\frac{1}{4\pi R}\cdot d\mathbf{S}', \tag{24}$$

subject to boundary conditions

$$\psi_0 = \psi \quad \text{on} \quad \sigma \tag{25}$$

and

$$\rho\partial\psi_0/\partial n = \rho_0\partial\psi/\partial n \quad \text{on} \quad \sigma. \tag{26}$$

Similarly, let χ_0 and χ be determined by

$$\chi_0(\mathbf{r}) = \oint \frac{1}{4\pi R}\nabla'\chi_0(\mathbf{r}')\cdot d\mathbf{S}' - \oint \chi_0(\mathbf{r}')\nabla'\frac{1}{4\pi R}\cdot d\mathbf{S}' \tag{27}$$

and

$$\chi(\mathbf{r}) = i\mathbf{k}_e\cdot\mathbf{r} - \oint \frac{1}{4\pi R}\nabla'\chi(\mathbf{r}')\cdot d\mathbf{S}' + \oint \chi(\mathbf{r}')\nabla'\frac{1}{4\pi R}\cdot d\mathbf{S}', \tag{28}$$

with boundary conditions

$$\chi_0=\chi \quad \text{on} \quad \sigma \tag{29}$$

and

$$\rho\partial\chi_0/\partial n=\rho_0\partial\chi/\partial n \quad \text{on} \quad \sigma. \tag{30}$$

Superposition then yields

$$p_0=\psi_0+\chi_0 \tag{31}$$

and

$$p=\psi+\chi. \tag{32}$$

By Eqs. 27 and 28 the potential χ_0 is due to an applied electric field $-i\mathbf{k}_e$, and by Eq. 19, χ_0 must have the property

$$\int \chi_0 dV=0 \tag{33}$$

in order that the applied field not contribute to the net charge q. Condition 33 can always be satisfied by a strategic choice of origin and may be regarded simply as a definition of the dimension L. Thus we may always write

$$q=\frac{\beta}{\beta_0}k^2\int\frac{\psi_0 dV}{4\pi}, \tag{34}$$

which means that the monopole coefficient q is determined independently by Eqs. 23 and 24. In the limit $\rho_0/\rho\rightarrow 0$, boundary condition 26 requires also that

$$\psi_0=\text{const.} \tag{35}$$

Thus we may also write

$$\boldsymbol{\mu}=-\left(\frac{\rho}{\rho_0}-1\right)\int\frac{\nabla\chi_0 dV}{4\pi}, \tag{36}$$

which means that the dipole coefficient $\boldsymbol{\mu}$ is determined independently by Eqs. 27 and 28.

The property $\psi_0=$const. has the additional consequence that Eqs. 23 and 24 determine the potential of a polarization-free dielectric replica of the scatterer, on which excess surface charge is distributed as if it were on a conducting replica. Therefore we may write

$$q=C\phi, \tag{37}$$

in which C is the capacitance (in electrostatic units) of a conducting replica of the scatterer and ϕ is the potential difference between this conductor and infinitely distant points. Equation 24 yields $\psi\rightarrow 1+ikq=1+ikC\phi$ for $r\rightarrow\infty$, and therefore the value $\psi=1+\phi+ikC\phi$ at the surface of the scatterer, whereupon Eq. 25 yields

$$\psi_0=1+\phi+ikC\phi. \tag{38}$$

From Eqs. 34, 35 and 37 we obtain

$$\psi_0=(\omega_r^2/\omega^2)\phi, \tag{39}$$

in which we have defined

$$\omega_r=(4\pi\beta_0 C/\rho V)^{\frac{1}{2}}, \tag{40}$$

and this combines with Eq. 38 to determine

$$\phi=(\omega_r^2/\omega^2-1-ikC)^{-1}. \tag{41}$$

Equations 37 and 39 have the consequence that $\psi_0=$const. satisfies Eq. 23 for frequencies $\omega\lesssim\omega_r$ in the limit $mkC\rho_0/\rho\rightarrow 0$. This demonstrates the self-consistency of our solution for an infinitely dilute resonance scatterer.[5]

Given the capacitance C, we now substitute $q=C\phi$ into Eq. 22 and put $\beta_0/\beta\ll 1$. Given the polarizability dyadic $\mathbf{P}$ (in electrostatic units) of the dielectric replica of the scatterer, the dipole moment $\boldsymbol{\mu}$ due to the applied electric field $-i\mathbf{k}_e$ is

$$\boldsymbol{\mu}=\mathbf{P}\cdot(-i\mathbf{k}_e), \tag{42}$$

which we also substitute into Eq. 22 and put $\rho_0/\rho\ll 1$. Thus if the quantities C and $\mathbf{P}$ are known, either by computation or from electrical measurements, the scattering amplitude for a dilute gaseous scatterer in a liquid medium is

$$f=\frac{C}{\omega_r^2/\omega^2-1-ikC}-\mathbf{k}_s\cdot\mathbf{P}\cdot\mathbf{k}_e. \tag{43}$$

This result exhibits two features that are not found in expression 2: (a) The axis of symmetry of the dipole term need not be parallel to the exciting wave vector $\mathbf{k}_e$, and (b) the monopole term contains a resonance denominator in which ω_r plays the role of resonance frequency.[3] Since $C=a$ and $V=\frac{4}{3}\pi a^3$ for a sphere, the resonance frequency computed from Eq. 40 is in agreement with Eq. 1. In the Appendix, it is shown by direct calculation that Eq. 40 also yields the correct resonance frequency for a prolate spheroid.

The polarizability dyadic for an ellipsoid illustrates how asymmetry of scattering arises as a second-order effect in the density contrast. In a coordinate system the axes of which are the principal axes of $\mathbf{P}$, the components of $\mathbf{P}$ are[6]

$$P_{mn}=\delta_{mn}(\rho/\rho_0-1)[1+(\rho/\rho_0-1)\gamma_m]^{-1}V/4\pi, \tag{44}$$

[5] It also indicates that we must have $mkC\rho_0/\rho\ll 1$ as well as $\rho_0/\rho\ll 1$ in order reasonably to assume that the solution is a valid approximation for a resonance scatterer of finite density. Since it seems improbable that mkC would exceed unity in a situation of interest, we assume that $\rho_0/\rho\ll 1$ is the stronger condition.

[6] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Co., New York, 1941), p. 212.

the quantities γ_m being dimensionless geometrical parameters for which $0 \leqslant \gamma_m \leqslant 1$. Resonance scattering is a second-order effect in the bulk modulus contrast, which is now investigated in some detail.

III. RESONANCE SCATTERING

By resonance, we mean that there exists a frequency interval for which kL is small in which the total scattering cross section Q exhibits a prominent maximum. The definition

$$Q = \int |f|^2 d\Omega, \tag{45}$$

where $d\Omega$ is an element of solid angle, together with Eq. 43, yields

$$Q = Q_1 + Q_2, \tag{46}$$

the sum of a monopole part

$$Q_1 = \frac{4\pi C^2}{(\omega_r^2/\omega^2 - 1)^2 + k^2 C^2} \tag{47}$$

and a dipole part

$$Q_2 = \tfrac{4}{3}\pi k^2 (\mathbf{P} \cdot \mathbf{k}_e)^2. \tag{48}$$

In order for resonance scattering to occur, first of all, the value of Q_1 for $\omega = \omega_r$ must greatly exceed the value at frequencies well above ω_r. In addition, the value of Q_1 must greatly exceed that of Q_2 for frequencies in a neighborhood of ω_r.

The first condition is met if the quantity $k_r^2 C^2$ is small, k_r being the value of k for $\omega = \omega_r$. To see what this means physically, it is convenient to define a shape factor,

$$F = 4\pi C^3 / 3V, \tag{49}$$

which has the value unity for a spherical scatterer. By a theorem of electrostatics—that of all conducting surfaces enclosing a given volume, the sphere is the surface of minimum capacitance[7]—the shape factor has the property

$$F \geqslant 1. \tag{50}$$

Since

$$k_r^2 C^2 = 3F\beta_0/\beta, \tag{51}$$

we conclude that resonance scattering actually occurs only if the ambient pressure is sufficiently low ($\beta_0 \ll \beta$) and only if the shape factor is sufficiently near unity that $3F\beta_0/\beta \ll 1$.

The relative importance of the dipole cross section for frequencies in a neighborhood of ω_r may be estimated using the polarizability dyadic for an ellipsoid for $\rho_0/\rho \to 0$. Except for a needlelike scatterer ($\gamma_1 = 0$), the components of $\mathbf{P}$ in this limit are

$$P_{mn} = (\delta_{mn}/\gamma_m) V / 4\pi. \tag{52}$$

Thus Q_2 is of order $4\pi C^2 (\omega/\omega_r)^4 (\beta_0/\beta)^2$ when ρ_0/ρ is small, whereas (except for frequencies well below ω_r) Q_1 is at least of order $4\pi C^2$. It seems reasonable, then, that $Q_1/Q_2 \gg 1$ quite generally when $\beta_0/\beta \ll 1$, for frequencies within an octave or so of ω_r. This is true also for all lower frequencies.

There are two classes of conditions under which our solution fails for frequencies as great as ω_r. (a) When L is sufficiently large as compared to C, it can happen that $k_r L = (3F\beta_0/\beta)^{\frac{1}{2}} L/C$ is not small even if $3F\beta_0/\beta \ll 1$. It appears that this type of failure occurs only for a needle-like scatterer. For example, L/C increases without limit as the eccentricity of a prolate spheroid increases but does not exceed $\pi/2$ for an oblate spheroid. (b) When the index of refraction is sufficiently large, it can happen that $\rho_0/\rho = m^2 \beta_0/\beta$ is not small even if $3F\beta_0/\beta \ll 1$. Since m may be as large as about 10 for certain combinations of gaseous scatterer and liquid external medium, this type of failure might seem to limit the applicability of our solution rather seriously. However, m is typically considerably smaller than 10 (the value for an air bubble in water is about $4\frac{1}{2}$). Moreover, the manner in which the assumption of small ρ_0/ρ is employed suggests that the condition need not be satisfied strongly in order for our solution to be a good approximation. Thus it appears that gaseous objects for which $3F\beta_0/\beta \ll 1$ are resonance scatterers of sound, except for scatterer shapes and possibly for material contrasts that do not often occur in practical situations, and that

$$Q = \frac{4\pi C^2}{(\omega_r^2/\omega^2 - 1)^2 + k^2 C^2} \tag{53}$$

is an adequate approximation for the resonance cross section.

Equation 53 does not, of course, account for viscous or thermal dissipation. Nor have the presence of multiple scatterers, discontinuities of the medium, or containment of the gas by a material that can support a shear stress been taken into account. The importance of such complications, and the detail in which they should be assessed, will depend upon the application at hand.

[7] G. Pólya and G. Szegö, *Isoperimetric Inequalities in Mathematical Physics* (Princeton University Press, Princeton, N. J., 1951), pp. vi and 8.

Appendix A

Let a prolate spheroid have major semidiameter a, minor semidiameter b, and interfocal distance d; and define $h=\frac{1}{2}kd$ and $\xi=(1-b^2/a^2)^{-\frac{1}{2}}$. The mode of vibration of the spheroid that yields monopole resonance scattering is describable in terms of the "radial" Mathieu functions, $je_{oo}(h,\xi)$ and $ne_{oo}(h,\xi)$. The lowest-order approximation in the small quantity $h\xi$ that retains a finite derivative is[A1]

$$je_{oo}(h,\xi)=1-\tfrac{1}{6}h^2\xi^2, \tag{A1}$$

$$ne_{oo}(h,\xi)=-h^{-1}\coth^{-1}\xi. \tag{A2}$$

For a plane wave exciting pressure of unit amplitude, the partial exciting pressure that drives this mode is

$$p_e=je_{oo}(h,j). \tag{A3}$$

The partial scattered and internal pressures are, respectively,

$$p_s=Ahe_{oo}(h,\xi) \tag{A4}$$

and

$$p_0=Bje_{oo}(h_0,\xi), \tag{A5}$$

[A1] Ref. 1, p. 1510.

in which $he_{oo}=je_{oo}+ine_{oo}$ and $h_o=\frac{1}{2}k_od$. The constants A and B are to be determined by the boundary conditions

$$p_s+p_e=p_0 \tag{A6}$$

and

$$\frac{\partial p_s}{\partial\xi}+\frac{\partial p_e}{\partial\xi}=\frac{\rho\partial p_0}{\rho_0\partial\xi}.$$

On putting $h\xi=ka$ and $\xi^2-1=\xi^2b^2/a^2$, the approximations $k^2a^2\ll1$, $k_o^2a^2\ll1$, and $\beta_o/\beta\ll1$ yield

$$A=\frac{ika/\xi\coth^{-1}\xi}{\omega_r^2/\omega^2-1-ika/\xi\coth^{-1}\xi}, \tag{A7}$$

in which

$$\omega_r=(3\beta_0/\rho b^2\coth^{-1}\xi)^{\frac{1}{2}} \tag{A8}$$

is the resonance frequency. Since $V=\frac{4}{3}\pi ab^2$ and $C=a/\xi\coth^{-1}\xi$ for a prolate ellipsoid,[A2] Eq. A8 is in agreement with Eq. 40. The cross section $Q=4\pi|A|^2/k^2$ computed from Eq. A7 is in agreement with Eq. 53.

[A2] Ref. 1, p. 1308.

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RESONANCE SCATTERING OF SOUND BY A SMALL GASEOUS OBJECT OF ARBITRARY FORM by M. S. Steinberg. Volume 41. PP. 1352-1357. March 1968. Contract Nonr-1367(00) NR 261-102.

The acoustic scattering amplitude for a dilute gaseous object of arbitrary form in a liquid medium is determined self-consistently for the case of linear dimension small as compared to the wavelength. A scatterer of bulk modules β_o and density ρ_o in a medium of bulk modulus β and density ρ may exhibit monopole resonance scattering at a frequency $\omega_r = (4\pi\beta_o C/\rho V)^{1/2}$, in which V is the volume and C is the capacitance in electrostatic units of a conducting replica of the scatterer. The criterion for occurrence of the resonance phenomenon is $3F\beta_o/\beta \ll 1$, in which the shape factor $F = 4\pi C^3/3V \geq 1$ is minimum for a sphere. Dipole scattering is given in terms of the polarizability dyadic of a nonconducting replica of dielectric constant ρ/ρ_o, and is negligibly small in a neighborhood of the resonance frequency.

1. Scattering
2. Acoustics
3. Gaseous Meteors

I. Steinberg, M. S.

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